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# Integral Coefficients for One-Loop Amplitudes

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Abstract: We present a set of algebraic functions for evaluating the coefficients of the scalar integral basis of a general one-loop amplitude. The functions are derived from unitarity cuts, but the complete cut-integral procedure has been carried out in generality so that it never needs to be repeated. Where the master integrals are known explicitly, the results here can be used as a black box with tree-level amplitudes as input and one-loop amplitudes as output.

Keywords: NLO Computations, QCD.

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## 1. Introduction

Observations of new physics at the Large Hadron Collider will come from the analysis of many scattering processes with complex final states. It is essential to prepare the theoretical groundwork by performing precision calculations of Standard Model production processes incorporating at least next-to-leading order QCD corrections. This effort requires efficient algorithms for computing multi-leg one-loop amplitudes.

The unitarity method introduced in [1] is designed to compute amplitudes by applying a unitarity cut to an amplitude on one hand, and its expansion in a basis of master integrals on the other [2, 3, 4, 5]. From knowledge of the basis and the general structure of the coefficients in the expansion, the coefficients can be constrained.

The holomorphic anomaly [6] reduces the problem of phase space integration to one of algebraic manipulation, namely evaluating residues of a complex function. By applying this operation within the unitarity method, coefficients can be extracted systematically. The reason this is possible is that the unitarity cuts of master integrals are uniquely identifiable as analytic expressions. Accordingly, a method was introduced to evaluate any finite four-dimensional unitarity cut and systematically derive compact expressions for the coefficients [7, 8]. The evaluation was carried out in the context of the spinor formalism [9, 10, 11, 12, 13, 14]. In [15], we wrote down these general, compact formulas for master integral coefficients.

The main purpose of the present paper is to improve upon those formulas in two respects. First, the coefficients were written as residues of the explicit formulas in [15]. Identifying the residue of a function simply involves performing a series expansion, but within the spinor formalism, this expansion is not very transparent in the case of multiple poles. Additional instructions were given to aid in automatizing this step. Here, our formulas will be given in terms of a truncated series expansion in a single scalar variable. Second, the starting point of the formulas of [15], from which to take input data, was the result of some spinorial manipulations of the initial cut integrand. Here, the input data are determined directly from the initial cut integrand, as assembled from tree-level amplitudes.

In this paper, we have thus eliminated the need for applying any analytic spinor identities. Programming the final formulas is completely straightforward. The values of the coefficients are of course identical to those from the formulas of our previous paper. Other general expressions for coefficients derived from unitarity cuts and generalized unitarity cuts of one-loop amplitudes have been given in [16, 17, 18, 19, 20].

Starting from analytic expressions for color-ordered tree amplitudes, we set up the unitarity cut integral. If $K$ is the momentum in the unitarity cut, then the two cut propagators can be denoted by $p$ and $p-K$. See Figure 1. In terms of its dependence on the loop momentum $p$, the cut

integral is a sum of terms of the following form:

$$C = c \int d^{4-2\epsilon}p \, \frac{\prod_{i=1}^{m}(-2p\cdot P_i)}{\prod_{j=1}^{k}(p-K_j)^2} \delta^{(+)}(p^2)\delta^{(+)}((p-K)^2) \tag{1.1}$$

Here, $c$ is the prefactor independent of $p$ (but may depend on $\mu^2$ (or $u$ discussed below) in our dimension regularization) , and the values of the momentum vectors $K_j$ are sums of momenta of cyclically adjacent external particles. We work in the four-dimensional helicity scheme, so that all external momenta $K_i$ are 4-dimensional and only the internal momentum $p$ is $(4-2\epsilon)$-dimensional. Hence we decompose the loop momentum as [21, 22]

$$p = \widetilde{\ell} + \vec{\mu}, \tag{1.2}$$

where $\widetilde{\ell}$ is 4-dimensional and $\vec{\mu}$ is $(-2\epsilon)$-dimensional, and we further define the extra-dimensional parameter $u$ by

$$u = \frac{4\mu^2}{K^2}. \tag{1.3}$$

Let us then define the following four-vectors:

$$\begin{aligned} Q_j &= -(\sqrt{1-u})K_j + \frac{K_j^2 - (1-\sqrt{1-u})(K_j\cdot K)}{K^2}K, \\ R_i &= -(\sqrt{1-u})P_i - \frac{(1-\sqrt{1-u})(P_i\cdot K)}{K^2}K. \end{aligned} \tag{1.4}$$

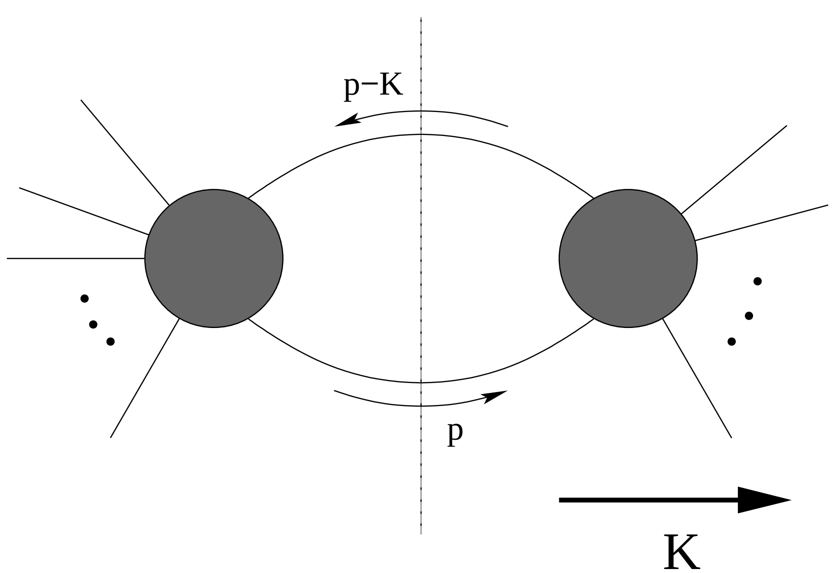


**Figure 1:** Representation of the cut integral. $K$ is the sum of external momenta on one side of the cut.

In terms of these momentum vectors, the cut integral may be expressed as

$$C = c \int_0^1 du\ u^{-1-\epsilon} \int \langle \ell\ d\ell\rangle\,[\ell\ d\ell](\sqrt{1-u})\frac{(K^2)^{n+1}}{\langle\ell|K|\ell]^{n+2}}\frac{\prod_{i=1}^{n+k}\langle\ell|R_i|\ell]}{\prod_{j=1}^{k}\langle\ell|Q_j|\ell]} \tag{1.5}$$

where we have set $n = m-k$, and $|\ell\rangle$ and $|\ell]$ are homogeneous spinors. This follows from the basic steps of spinor integration, which are reviewed in Appendix A but are not needed to apply our formulas for coefficients. The point is that the only thing we need to do is treat a general integrand of the form

$$(\sqrt{1-u})(K^2)^{1+n}\frac{1}{\langle\ell|K|\ell]^{n+2}}\frac{\prod_{j=1}^{k+n}\langle\ell|R_j|\ell]}{\prod_{i=1}^{k}\langle\ell|Q_i|\ell]}\ . \tag{1.6}$$

The result of this integration is the subject of this paper. In terms of the vectors defined in (1.4) from the initial data of (1.1), and the two integers $k$ and $n$, we give formulas for the four-dimensional coefficients. For renormalizable theories, $n \leq 2$. Terms with $n \leq -2$ contribute to box integrals only; terms with $n = -1$ contribute to triangle and box integrals; and terms with $n \geq 0$ contribute to bubble, triangle and box integrals. To proceed to the full $d$-dimensional coefficients, including those for pentagons, one would perform the final integral over $u$ with the recursion and reduction formulas of [23, 24].

We wish to remark on a few features of our formulas.

- Our starting point is the most general expression in field theory with a unitarity cut of a one-loop amplitude. Particles can be massless or massive, although in this paper we focus on massless propagators. Generalization to massive propagators should be straightforward. The propagator can be a scalar, fermion, or vector, as long as the proper degrees of freedom are accounted for.[1]

- The analytic formulas for tree amplitudes needed as input should be free of unphysical singularities involving the loop momentum $p$, so that the form of (1.1) is apparent. This is especially important when using on-shell recursion relations to derive tree-level amplitudes. Using Feynman diagrams or Berends-Giele recursion [25] to get tree-level expressions automatically circumvents this problem.

- With our formulas, we can calculate any particular coefficient directly without reference to other coefficients.

[1]One way to do this is to use Feynman diagrams to write down the full one-loop integrand expression, then multiply by $(p^2-m_1^2)^{-1}((p-K)^2-m_2^2)^{-1}$ along with the two delta-functions $\delta(p^2-m_1^2)\delta((p-K)^2-m_2^2)$, and use these to set e.g. $p^2 = m_1^2$ in the integrand. In this way, even if propagators are fermions or vectors, we have counted everything.

- Our formulas can easily be used to obtain the 4-dimensional part of the coefficients only,[2] by taking the limit $u \to 0$ in (1.4). However, to be sure that all intermediate formulas will be well-defined, it is safest to take this limit at the end of the calculation. If we do wish to set $u \to 0$ at the beginning beginning, some care must be taken, as discussed in Section 3.2.

- Our formulas work for any $n$, although for renormalizable theories we will have $n \leq 2$. But if we consider (super)gravity or use a bad gauge choice, then we would have $n > 2$.

- In spinor notation, we will find factors of the form $\langle a|p|b]$ in the numerator. This can be rewritten as $-2p \cdot P$ with $P = \lambda_a \widetilde{\lambda}_b$. So $P_i$ can take complex values in (1.4).

The formulas are presented in Section 2. In Section 3, we present an example of a 4-dimensional unitarity cut in one helicity configuration of the six-gluon amplitude. In Section 4, we give examples from the four-gluon amplitude but keep the full $d$-dimensional dependence. In Section 5, we discuss future applications and comparisons to other techniques. Appendix A reviews the first steps in spinor integration which form the basis of the derivation of our coefficients. Most details of the derivation of our formulas are presented in Appendix B. Appendix C contains the formulas for triangle coefficients after the truncated series expansion has been carried out explicitly, although direct the use of the result in Section 2 is likely to be simpler.

## 2. Coefficients of box, triangle and bubble integrals

Here we give the results for the box, triangle, and bubble coefficients in the unitarity cut defined by the momentum $K$, starting from the integrand (1.1) without the prefactor $c$, and using the definitions (1.4). Our convention is that the $n$-point scalar function is defined by [3]

$$I_n = i(4\pi)^{(4-2\epsilon)/2} \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{1}{p^2(p-K_1)^2(p-K_1-K_2)^2 ... (p-\sum_{j=1}^{n-1} K_j)^2}. \tag{2.1}$$

The spinor notation we use here, which may differ from other conventions, is defined as follows. For a four-vector $k_i$ satisfying $k_i^2 = 0$,

$$\lambda_i \equiv u_+(k_i), \qquad \widetilde{\lambda}_i \equiv u_-(k_i), \tag{2.2}$$

thus we have the following inner products:

$$\langle i\ j\rangle = \left\langle i^-|j^+\right\rangle = \overline{u}_-(k_i)u_+(k_j), \qquad [i\ j] = [i^+|j^-] = \overline{u}_+(k_i)u_-(k_j) \tag{2.3}$$

[2] That is to say, neglecting possible rational terms which can be calculated by other methods, for example by the recursive techniques of [26, 27, 28, 29, 30] or the specialized diagrammatic reductions of [31, 32].

[3] We omit the prefactor $(-1)^{n+1}$ that is common elsewhere in the literature [3, 4].

Note that in this paper we use "twistor" sign conventions, so that

$$2k_i \cdot k_j = \langle i\ j\rangle\, [i\ j] \tag{2.4}$$

which differs from the standard QCD convention by a minus sign for each spinor product $[i\ j]$. Our definitions implying the following relations:

$$\langle i|P|j] = \overline{u}_-(k_i)\ \not{P}\ u_-(k_j), \qquad \langle i|P_1 P_2|j\rangle = \overline{u}_-(k_i)\ \not{P}_1\ \not{P}_2\ u_+(k_j) \tag{2.5}$$

The full $d$-dimensional amplitude will generically include pentagons in the basis. The identification of pentagon coefficients has already been described in [24]. The operation occurs in the final integral over $u$. Since our purpose here is to give the results of the 4-dimensional integration, we will not now comment any further on pentagons. In cases involving massive species, tadpole integrals can also arise. Unitarity methods cannot detect these. However, we expect that it will be possible to fix tadpole coefficients from other considerations, such as a heavy mass limit [33].

### 2.1 Box coefficients

A box integral is identified by the two cut propagators plus two additional ones. Following the setup of the previous section, denote the two additional momenta associated to a box by $K_r$ and $K_s$. Then, define the vectors $Q_r$ and $Q_s$ as in (1.4). From these two vectors, we construct two null vectors $P_{sr,1}$ and $P_{sr,2}$ as follows:[4]

$$\begin{aligned} \Delta_{sr} &= (2Q_s \cdot Q_r)^2 - 4Q_s^2 Q_r^2 \\ P_{sr,1} &= Q_s + \left( \frac{-2Q_s \cdot Q_r + \sqrt{\Delta_{sr}}}{2Q_r^2} \right) Q_r \\ P_{sr,2} &= Q_s + \left( \frac{-2Q_s \cdot Q_r - \sqrt{\Delta_{sr}}}{2Q_r^2} \right) Q_r \end{aligned} \tag{2.6}$$

Then, the box coefficient with momenta $K, K_r, K_s$ is given by

$$C[Q_r, Q_s, K] = \frac{(K^2)^{2+n}}{2} \left( \frac{\prod_{j=1}^{k+n} \langle P_{sr,1}|R_j|P_{sr,2}]}{\langle P_{sr,1}|K|P_{sr,2}]^{n+2} \prod_{t=1,t\neq i,j}^{k} \langle P_{sr,1}|Q_t|P_{sr,2}]} + \{P_{sr,1} \leftrightarrow P_{sr,2}\} \right). \tag{2.7}$$

[4] Here we see that the formula we give is ill-defined in special cases where $Q_r^2 = 0$. This case can arise if we have set $u$ to zero to find a 4-dimensional coefficient, and the external momentum $K_r$ is null. However, there is no difficulty with the underlying method. For a given box, $Q_r$ and $Q_s$ can be exchanged. Clearly, if both $K_r$ and $K_s$ are null, we can simply take $P_{sr,1} = Q_s, P_{sr,2} = Q_r$. In practice, this problem can always be avoided by keeping $u$ finite until the end of the calculation.

## 2.2 Triangle coefficients

A triangle integral is identified by the two cut propagators plus one additional one. If the additional momentum variable is $K_s$, then define the vector $Q_s$ as in (1.4). Now construct two null vectors $P_{s,1}$ and $P_{s,2}$ as follows:

$$\begin{aligned}\Delta_s &= (2Q_s \cdot K)^2 - 4Q_s^2 K^2 \\ P_{s,1} &= Q_s + \left(\frac{-2Q_s \cdot K + \sqrt{\Delta_s}}{2K^2}\right) K \\ P_{s,2} &= Q_s + \left(\frac{-2Q_s \cdot K - \sqrt{\Delta_s}}{2K^2}\right) K\end{aligned} \tag{2.8}$$

Then, the triangle coefficient with momenta $K, K_s$ is given by

$$\begin{aligned}C[Q_s, K] = &\frac{(K^2)^{1+n}}{2} \frac{1}{(\sqrt{\Delta_s})^{n+1}} \frac{1}{(n+1)!\,\langle P_{s,1}\ P_{s,2}\rangle^{n+1}} \\ &\times \frac{d^{n+1}}{d\tau^{n+1}} \left( \frac{\prod_{j=1}^{k+n} \langle P_{s,1} - \tau P_{s,2} | R_j Q_s | P_{s,1} - \tau P_{s,2}\rangle}{\prod_{t=1, t\neq s}^{k} \langle P_{s,1} - \tau P_{s,2} | Q_t Q_s | P_{s,1} - \tau P_{s,2}\rangle} + \{P_{s,1} \leftrightarrow P_{s,2}\} \right)\Bigg|_{\tau=0} .\end{aligned} \tag{2.9}$$

In practice, the multiple derivative is easy to perform in a symbolic manipulation program, either analytically or numerically, and we believe this is an efficient presentation of the coefficient. However, we have found closed expressions, and these are given in Appendix C for $n \leq 2$.

If $n \leq -2$, then the coefficient is simply zero.

## 2.3 Bubble coefficients

Every unitarity cut singles out a unique bubble integral. However, in our derivation, bubble and triangle integrals are related, so the following formulas still require the quantities defined in (2.8). We further introduce two arbitrary *real* null vectors,[5] $\eta$ and $\tilde{\eta}$, and their associated spinors. These null vectors must, however, be chosen generically: they should not coincide with other momentum variables.

The coefficient of the bubble integral with momentum $K$ is given by

$$C[K] = (K^2)^{1+n} \sum_{q=0}^{n} \frac{(-1)^q}{q!} \frac{d^q}{ds^q} \left( \mathcal{B}_{n,n-q}^{(0)}(s) + \sum_{r=1}^{k} \sum_{a=q}^{n} \left( \mathcal{B}_{n,n-a}^{(r;a-q;1)}(s) - \mathcal{B}_{n,n-a}^{(r;a-q;2)}(s) \right) \right)\Bigg|_{s=0} , \tag{2.10}$$

[5]The reality condition is important. These vectors should be physical momenta of massless particles.

where

$$\mathcal{B}_{n,t}^{(0)}(s) \equiv \frac{d^n}{d\tau^n}\left(\frac{1}{n![\eta|\widetilde{\eta}K|\eta]^n}\frac{(2\eta\cdot K)^{t+1}}{(t+1)(K^2)^{t+1}}\frac{\prod_{j=1}^{n+k}\langle\ell|R_j(K+s\eta)|\ell\rangle}{\langle\ell\ \eta\rangle^{n+1}\prod_{p=1}^{k}\langle\ell|Q_p(K+s\eta)|\ell\rangle}|_{|\ell\rangle\to|K-\tau\widetilde{\eta}|\eta]}\right)\Bigg|_{\tau=0}, \tag{2.11}$$

$$\begin{aligned}\mathcal{B}_{n,t}^{(r;b;1)}(s) \equiv & \frac{(-1)^{b+1}}{b!\sqrt{\Delta_r}^{b+1}\langle P_{r,1}\ P_{r,2}\rangle^b}\frac{d^b}{d\tau^b}\left(\frac{1}{(t+1)}\frac{\langle P_{r,1}-\tau P_{r,2}|\eta|P_{r,1}]^{t+1}}{\langle P_{r,1}-\tau P_{r,2}|K|P_{r,1}]^{t+1}}\right.\\ & \times\left.\frac{\langle P_{r,1}-\tau P_{r,2}|Q_r\eta|P_{r,1}-\tau P_{r,2}\rangle^b\prod_{j=1}^{n+k}\langle P_{r,1}-\tau P_{r,2}|R_j(K+s\eta)|P_{r,1}-\tau P_{r,2}\rangle}{\langle P_{r,1}-\tau P_{r,2}|\eta K|P_{r,1}-\tau P_{r,2}\rangle^{n+1}\prod_{p=1,p\neq r}^{k}\langle P_{r,1}-\tau P_{r,2}|Q_p(K+s\eta)|P_{r,1}-\tau P_{r,2}\rangle}\right)\Bigg|_{\tau=0}, \end{aligned}\tag{2.12}$$

$$\begin{aligned}\mathcal{B}_{n,t}^{(r;b;2)}(s) \equiv & \frac{(-1)^{b+1}}{b!\sqrt{\Delta_r}^{b+1}\langle P_{r,1}\ P_{r,2}\rangle^b}\frac{d^b}{d\tau^b}\left(\frac{1}{(t+1)}\frac{\langle P_{r,2}-\tau P_{r,1}|\eta|P_{r,2}]^{t+1}}{\langle P_{r,2}-\tau P_{r,1}|K|P_{r,2}]^{t+1}}\right.\\ & \times\left.\frac{\langle P_{r,2}-\tau P_{r,1}|Q_r\eta|P_{r,2}-\tau P_{r,1}\rangle^b\prod_{j=1}^{n+k}\langle P_{r,2}-\tau P_{r,1}|R_j(K+s\eta)|P_{r,2}-\tau P_{r,1}\rangle}{\langle P_{r,2}-\tau P_{r,1}|\eta K|P_{r,2}-\tau P_{r,1}\rangle^{n+1}\prod_{p=1,p\neq r}^{k}\langle P_{r,2}-\tau P_{r,1}|Q_p(K+s\eta)|P_{r,2}-\tau P_{r,1}\rangle}\right)\Bigg|_{\tau=0} \end{aligned}\tag{2.13}$$

Before ending this section, we want to make an additional remark. The derivation of these formulas, given in Appendix B, involved reducing the degree of $\widetilde{\lambda}$ in both numerators and denominators. However, we could just as well choose to reduce the degree of $\lambda$ instead. In this case we would get formulas with the following replacement: $|\star\rangle \to |\star]$ and $|\star] \to |\star\rangle$. These two sets of formulas are equivalent to each other in the case $u \neq 0$. But if we naively set $u = 0$ from the beginning, it is possible that one of the two sets of formulas will break down. Such an example will be seen in Section 3.2.

## 3. An example from the six-gluon amplitude

In this section we test our formulas by computing some coefficients from a one-loop partial amplitude with six external gluons and an adjoint scalar circulating in the loop. These contribute to the full six-gluon amplitude in the spinor-helicity formalism in the context of the supersymmetric decomposition [34, 1, 35]. The box coefficient was first computed in [36], and the bubble coefficient was first computed in [8]. The two-mass-triangle coefficients have not appeared in this form before, because it is possible to modify the basis and eliminate the corresponding integrals, as described in [7].[6] In [7] it was shown that for gluon amplitudes, these coefficients are constrained by IR and UV divergences, and we use that relation here as a consistency check.

[6]In practice it is probably best to use the modified basis, because it is less divergent.

Here we set the dimensional parameter $u$ to zero from the start in order to work with simpler expressions. As described above, this simplification requires some care, and in fact we will see the consequences when we derive the triangle coefficients.

We choose the unitarity cut of the momentum $K \equiv k_4 + k_5 + k_6$ in the helicity configuration $(1^-2^-3^+4^-5^+6^+)$. The cut integral is

$$
\begin{aligned}
C_{123} &= \int d\mu\; 2A(\ell_1^-, 1^-, 2^-, 3^+, \ell_2^+)A((-\ell_2)^-, 4^-, 5^+, 6^+, (-\ell_1)^+) \\
&= \frac{2}{s_{456}[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle}\int d\mu\; \frac{\langle 4\ \ell_1\rangle^2\langle 4\ \ell_2\rangle[3\ \ell_1]^2[3\ \ell_2]}{\langle 6\ \ell_1\rangle[\ell_1\ 1]} \\
&= -\frac{2}{s_{456}\,[1\ 2]\,[2\ 3]\,\langle 4\ 5\rangle\,\langle 5\ 6\rangle}\int d\mu\; \frac{\langle 1|\ell|6]\,\langle 4|\ell|3]^3}{(\ell-k_6)^2(\ell+k_1)^2} \\
&\quad + \frac{2\,\langle 4|K|3]}{s_{456}\,[1\ 2]\,[2\ 3]\,\langle 4\ 5\rangle\,\langle 5\ 6\rangle}\int d\mu\; \frac{\langle 1|\ell|6]\,\langle 4|\ell|3]^2}{(\ell-k_6)^2(\ell+k_1)^2}
\end{aligned} \tag{3.1}
$$

So we have two terms, each with $k = 2$, and

$$K_1 = k_6, \qquad K_2 = -k_1,$$

so

$$Q_1 = -k_6, \qquad Q_2 = k_1.$$

In the first term $m = 4$, and in the second term $m = 3$. We have

$$R_1 = -\lambda_1\tilde{\lambda}_6, \qquad R_2 = R_3 = R_4 = -\lambda_4\tilde{\lambda}_3.$$

### 3.1 Box coefficient

Since $k = 2$, we see immediately that there can be only one nonvanishing box coefficient in this cut. We compute the null vectors $P_{sr,1}$ and $P_{sr,2}$ from the definitions in (2.6), and define the associated spinors as follows.[7]

$$
\begin{array}{lll}
P_{12,1} = Q_1 = -k_6 & \quad |P_{12,1}\rangle = |6\rangle & \quad |P_{12,1}] = -|6] \\
P_{12,2} = Q_2 = k_1 & \quad |P_{12,2}\rangle = |1\rangle & \quad |P_{12,2}] = |1]
\end{array}
$$

[7] We could just as well use

$$P_{21,1} = Q_2 = k_1,\ |P_{21,1}\rangle = |1\rangle\,,\ |P_{21,1}] = |1],$$

$$P_{21,2} = Q_1 = -k_6,\ |P_{21,2}\rangle = |6\rangle\,,\ |P_{21,2}] = -|6].$$

Applying (2.7) to the expressions under the integral signs in (3.1), we get

$$C[Q_1, Q_2, K] = \frac{s_{456}^{2+n}}{2}\left(\frac{(-1)^{n+2} s_{61} \langle 6\ 4\rangle^{n+1} [3\ 1]^{n+1}}{\langle 6|K|1]^{n+2}}\right)$$

Now attach the prefactors for each of the two terms. For the first term with $n = 2$:

$$-\frac{2}{s_{456}[1\ 2][2\ 3]\langle 4\ 5\rangle \langle 5\ 6\rangle} \times C[Q_1, Q_2, K] = -\frac{s_{456}^3 s_{61} \langle 6\ 4\rangle^3 [3\ 1]^3}{[1\ 2][2\ 3]\langle 4\ 5\rangle \langle 5\ 6\rangle \langle 6|K|1]^4}$$

For the second term with $n = 1$:

$$\frac{2\langle 4|K|3]}{s_{456}[1\ 2][2\ 3]\langle 4\ 5\rangle \langle 5\ 6\rangle} \times C[Q_1, Q_2, K] = -\frac{s_{456}^2 s_{61} \langle 6\ 4\rangle^2 [3\ 1]^2 \langle 4|K|3]}{[1\ 2][2\ 3]\langle 4\ 5\rangle \langle 5\ 6\rangle \langle 6|K|1]^3}$$

So the total coefficient of the box $(1|23|45|6)$, for the scalar contribution, is

$$-\frac{s_{456}^2 s_{61} \langle 6\ 4\rangle^2 [3\ 1]^2 \langle 6|K|3] \langle 4|K|1]}{[1\ 2][2\ 3]\langle 4\ 5\rangle \langle 5\ 6\rangle \langle 6|K|1]^4} \tag{3.2}$$

This agrees with the expression in [36] when we incorporate the usual factor of $2/(s_{456}s_{61})$.

### 3.2 Triangle coefficients

Since $k = 2$, we see immediately that there can be only two nonvanishing triangle coefficients in this cut.

For $Q_1$:

$$\begin{array}{lll} \sqrt{\Delta_1} = -s_{456} + s_{45} & & \\ P_{1,1} = -k_6 & |P_{1,1}\rangle = |6\rangle & |P_{1,1}] = -|6] \\ P_{1,2} = \frac{s_{456}-s_{45}}{s_{456}}(k_4 + k_5) - \frac{s_{45}}{s_{456}} k_6 & |P_{1,2}\rangle = \frac{K|6]}{s_{456}} & |P_{1,2}] = K\,|6\rangle \end{array}$$

For $Q_2$:

$$\begin{array}{lll} \sqrt{\Delta_2} = -s_{456} + s_{23} & & \\ P_{2,1} = k_1 & |P_{2,1}\rangle = |1\rangle & |P_{2,1}] = |1] \\ P_{2,2} = \frac{s_{23}}{s_{456}} k_1 - \frac{s_{456}-s_{23}}{s_{456}}(k_2 + k_3) & |P_{2,2}\rangle = \frac{K|1]}{s_{456}} & |P_{2,2}] = -K\,|1\rangle \end{array}$$

Let us first consider the triangle $(1|23|456)$, with momenta $K$ and $K_2$ $(Q_2)$. With the identity

$$\frac{d^{n+1}}{d\tau^{n+1}}\left(\frac{(a-\tau b)^{n+1}(-\tau)^{n+2}}{c-\tau d} + \frac{(b-\tau a)^{n+1}}{d-\tau c}\right)\Bigg|_{\tau\to 0} = \frac{(n+1)!(bc-ad)^{n+1}}{d^{n+2}}$$

we see that the formula (2.9) for triangle coefficients becomes

$$C[Q_2,K]=\frac{s_{456}^{1+n}}{2}\frac{\langle 1|K|1]\,\langle 4\ 6\rangle^{n+1}\,[3\ 1]^{n+1}}{\langle 6|K|1]^{n+2}}$$

Adding the $n=1$ and $n=2$ contributions and attaching the prefactors, we find that the total coefficient is

$$\begin{aligned}
&-\frac{2}{s_{456}[1\ 2][2\ 3]\,\langle 4\ 5\rangle\,\langle 5\ 6\rangle}\left(\frac{s_{456}^3}{2}\frac{\langle 1|K|1]\,\langle 4\ 6\rangle^3\,[3\ 1]^3}{\langle 6|K|1]^4}\right)\\
&+\frac{2\,\langle 4|K|3]}{s_{456}[1\ 2][2\ 3]\,\langle 4\ 5\rangle\,\langle 5\ 6\rangle}\left(\frac{s_{456}^2}{2}\frac{\langle 1|K|1]\,\langle 4\ 6\rangle^2\,[3\ 1]^2}{\langle 6|K|1]^3}\right)\\
&=\frac{s_{456}\,\langle 1|K|1]\,\langle 4|K|1]\,\langle 6|K|3]\,[3\ 1]^2\,\langle 4\ 6\rangle^2}{[1\ 2][2\ 3]\,\langle 4\ 5\rangle\,\langle 5\ 6\rangle\,\langle 6|K|1]^4}
\end{aligned}\tag{3.3}$$

Now consider the triangle (123|45|6), with momenta $K$ and $K_1$ ($Q_1$). A naive application of the formula (2.9) in the limit $u\to 0$ gives

$$C[Q_1,K]=0$$

because of the $R_1Q_1$ contraction in the numerator's product. But the triangle coefficient does not actually vanish! This is clear, because this triangle is related by conjugation and label permutation to the previous one. This is the degenerate case that we discussed at the end of the previous section. The reason is clear. From (3.1), we see that for $Q_2$, the pole is $[\ell\ 1]$, while for $Q_1$ it is $\langle \ell\ 6\rangle$. Since the formula given in previous section is obtained by writing total derivative in $[d\widetilde{\alpha}\ \partial_{\widetilde{\lambda}}]$, they are not suitable for pole $\langle \ell\ 6\rangle$. To deal with it we need to use the conjugate formula where we replace $|\star\rangle\to|\star]$ and $|\star]\to|\star\rangle$, i.e., writing a total derivative of the form $\langle d\alpha\ \partial_\lambda\rangle$. This will be a general rule in all 4-dimensional calculations with null poles. We want to emphasize that this situation will not arise if we keep $u\neq 0$ until the end.

After clarifying the subtle point we can continue our calculation. Either by taking the conjugate of the formula (2.10), which is

$$\frac{s_{123}^{n+1}}{2(n+1)!(\sqrt{\Delta_1})^{n+1}[P_1\ P_2]^{n+1}}\frac{d^{n+1}}{d\tau^{n+1}}\left(\frac{[P_1-\tau P_2|R_1Q_1|P_1-\tau P_2][P_1-\tau P_2|R_2Q_1|P_1-\tau P_2]^{n+1}}{[P_1-\tau P_2|Q_2Q_1|P_1-\tau P_2]}+\{P_1\leftrightarrow P_2\}\right),$$

or by directly applying the relabeling and conjugation to (3.3), we find

$$C[Q_1,K]=-\frac{s_{456}^{n+1}\,\langle 6|K|6]\,\langle 6\ 4\rangle^{n+1}\,[1\ 3]^{n+1}}{2\,\langle 6|K|1]^{n+2}}.$$

Adding the two terms from $n = 2$ and $n = 1$, with prefactors, we get

$$
\begin{aligned}
&-\frac{s_{456}\langle 6|K|6]\langle 6\ 4\rangle^2[1\ 3]^2}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle}(-s_{456}\langle 6\ 4\rangle[1\ 3]+\langle 4|K|3]\langle 6|K|1])\\
&= -\frac{s_{456}\langle 6|K|6]\langle 6\ 4\rangle^2[1\ 3]^2\langle 4|K|1]\langle 6|K|3]}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 6|K|1]^4}
\end{aligned}
\tag{3.4}
$$

**Consistency check:**

These particular triangle coefficients have not been isolated before, because two-mass triangles disappear in the modified integral basis proposed in [7]. We can now perform a consistency check based on the same identity that allowed the basis to be modified. Consider all the contributions to the divergence $(-s)^{-\epsilon}$, where $s = K^2$. In these example, there are exactly these two 2-mass triangles plus the single box from the previous subsection. The condition expressing the vanishing of this divergence is [8]

$$
\begin{aligned}
0 &= c_4^{2m\ h}\frac{2}{st} - \sum c_3^{2m}(s,t)\frac{1}{(-s)-(-t)} \qquad (3.5)\\
&= c_4^{2m\ h}\frac{2}{s_{456}s_{61}} - c_{[1|23|456]}\frac{1}{-s_{456}+s_{23}} - c_{[123|45|6]}\frac{1}{-s_{456}+s_{45}}\\
&= c_4^{2m\ h}\frac{2}{s_{456}s_{61}} + c_{[1|23|456]}\frac{1}{\langle 1|K|1]} - c_{[1|23|456]}\frac{1}{\langle 6|K|6]}.
\end{aligned}
$$

It is easy to see that this identity is satisfied by our coefficients given in (3.2),(3.3),(3.4).

### 3.3 Bubble coefficient

Let us choose $\eta = 3$ and $\tilde{\eta} = 4$. This choice gives somewhat simpler formulas; for example, $\mathcal{B}^{(0)}_{n,n-q}(s)$ and $\mathcal{B}^{(2;a-q;1)}_{n,n-a}(s)$ are identically zero, because there is a sufficiently high power of $\tau$ inside the derivative. We find

$$
\mathcal{B}^{(0)}_{n,n-q}(s) = 0
$$

$$
\begin{aligned}
\mathcal{B}^{(1;a-q;1)}_{n,n-a}(s) = &\frac{(-1)^a[6\ 3]^{a-q}}{\langle 6|K|6]^{n-q+2}(a-q)!}\frac{d^{a-q}}{d\tau^{a-q}}\Bigg(\frac{[3\ 6]^{n-a+1}}{s_{456}^{n+1-q}(n-a+1)}\\
&\times\frac{\tau^{a-q}(s_{456}[6|K+sk_3|6\rangle-\tau s[6|3|K|6])(s_{456}\langle 4\ 6\rangle-\tau\langle 4|K|6])^{n+1}}{(s_{456}\langle 3\ 6\rangle-\tau\langle 3|K|6])^q(s_{456}[1|K+sk_3|6\rangle-\tau[1|K+sk_3|K|6])}\Bigg)\Bigg|_{\tau=0}
\end{aligned}
$$

[8] The relative sign between box and triangle terms comes because our sign conventions for the master integrals (2.1) differ from those of [7].

$$\mathcal{B}^{(1;a-q;2)}_{n,n-a}(s) = \frac{(-1)^q \left[6\ 3\right]^{a-q}}{\langle 6|K|6]^{n-q+2}\,(a-q)!}\frac{d^{a-q}}{d\tau^{a-q}}\left(\frac{[3|K|6\rangle^{n-a+1}}{s_{456}^{n+1-q}(n-a+1)}\right.$$
$$\left.\times\ \frac{(s\,[6|3|K|6] - \tau s_{456}\,[6|K+sk_3|6\rangle)(\langle 4|K|6] - \tau s_{456}\,\langle 4\ 6\rangle)^{n+1}}{(\langle 3|K|6] - \tau s_{456}\,\langle 3\ 6\rangle)^q([1|K+sk_3|K|6] - \tau s_{456}\,[1|K+sk_3|6\rangle)}\right)\Bigg|_{\tau=0}$$

$$\mathcal{B}^{(2;a-q;1)}_{n,n-a}(s) = 0$$

$$\mathcal{B}^{(2;a-q;2)}_{n,n-a}(s) = \frac{(-1)^q \left[1\ 3\right]^{a-q}}{\langle 1|K|1]^{n-q+1}\,(a-q)!}\frac{d^{a-q}}{d\tau^{a-q}}\left(\frac{[3|K|1\rangle^{n-a+1}}{s_{456}^{n+1-q}(n-a+1)}\right.$$
$$\left.\times\ \frac{(\langle 4|K|1] - \tau s_{456}\,\langle 4\ 1\rangle)^{n+1}}{(\langle 3|K|1] - \tau s_{456}\,\langle 3\ 1\rangle)^q(\langle 6|K|1] - \tau s_{456}\,\langle 6\ 1\rangle)}\right)\Bigg|_{\tau=0}$$

We then substitute these expressions into (2.10) and attach the prefactors. For $\mathcal{B}^{(2;a-q;2)}_{n,n-a}(s)$, in fact only the $q=0$ contributions matter, because the $s$-dependence has dropped out with our choice of $|\eta\rangle = |3\rangle$. We have checked numerically that the result agrees with the corresponding result derived by the technique of [8].[9]

## 4. A $d$-dimensional example: four gluons

In this section we illustrate the use of the formulas in Section 2 in the case of four gluons with a scalar propagating in the loop. These amplitudes were first given in [22]. Our notation and presentation here are more similar to [37] and especially [24], where these amplitudes were derived by newer techniques. Here we have verified that our results reproduce those in the literature.[10] We stop just before the final integral over $u$, which could in general be done by the techniques of [23, 24], developed in the context of $d$-dimensional unitarity [38, 22, 39, 40, 37]. Note therefore that the labels of "box, triangle, bubble" are used in the $d$-dimensional sense. The four-gluon amplitude is a nice test of our formulas, because this simple case is where they are most likely to break down, for example by a bad choice of $\eta$, as we shall see in the last configuration. We consider three of the four independent helicity configurations, since the fourth adds no new features.

For ease of presentation, we make use of the variable $z$ as given in (A.1).

[9] We were unable to confirm numerically the printed value of the corresponding coefficient in [8], so we repeated the calculation.

[10] The $-+-+$ amplitude was given the wrong overall sign in equation (4.31) of [24].

### 4.1 $(1^+, 2^+, 3^+, 4^+)$

The simplest helicity configuration is $(++++)$. The integrand for the cut $K = K_{12}$ is

$$\frac{2\mu^4[1\ 2][3\ 4]}{\langle 1\ 2\rangle \langle 3\ 4\rangle}\frac{1}{(p-k_1)^2(p+k_4)^2}$$

From this, by comparing with general formula (1.1) we have $m = 0$, $k = 2$, $K_1 = k_1, K_2 = -k_4$, thus $n \equiv m - k = -2$, so there are neither triangle nor bubble contributions. There is only one box coefficient. There are no vectors $P_i$. The expression inside the parentheses in (2.7) thus degenerates to 1, so we find

$$\frac{2\mu^4[1\ 2][3\ 4]}{\langle 1\ 2\rangle \langle 3\ 4\rangle}\frac{1}{2}(1+1) = \frac{2\mu^4[1\ 2][3\ 4]}{\langle 1\ 2\rangle \langle 3\ 4\rangle}. \tag{4.1}$$

### 4.2 $(1^-, 2^+, 3^+, 4^+)$

For this case we have two cuts $C_{12}$ and $C_{41}$. These two cuts are related to each other by symmetry, so we focus on cut $C_{41}$. The integrand is

$$-\frac{u[2\ 3]}{2\langle 2\ 3\rangle}\frac{\langle 1|p|4]^2}{(p-k_4)(p+k_3)^2}$$

For this case we have $m = k = 2$ so $n = m - k = 0$, $K_1 = k_4, K_2 = -k_3$, thus

$$P = \lambda_1\widetilde{\lambda}_4, \qquad R = R_1 = R_2 = -(1-2z)\lambda_1\widetilde{\lambda}_4$$
$$Q_1 = -zk_1 - (1-z)k_4, \qquad Q_2 = (1-z)k_3 + zk_2$$

**Box:** Using

$$Q_1^2 = Q_2^2 = \frac{u}{4}K_{41}^2, \quad 2Q_1\cdot Q_2 = -K_{12}^2 + \frac{u}{2}(K_{12}^2 - K_{13}^2)$$
$$\Delta_{12} = s^2(1-u)(1+u\frac{t}{s}), \quad s = K_{12}^2,\ t = K_{13}^2,$$

we have

$$-\frac{u[2\ 3]}{2\langle 2\ 3\rangle}\frac{[4|3|1\rangle^2(2+Au)}{2s_{13}^2}\frac{s_{41}^2}{2} = -\frac{s_{41}[2\ 3]^2[4\ 3]^2u(2+Au)}{8[1\ 3]^2}, \qquad A = \frac{s_{13}}{s_{12}}. \tag{4.2}$$

**Triangle:**

Formally, there are two triangles identified by the momenta $K_1$ and $K_2$. In this special example with only four external particles, they are, in fact, the *same* triangle. So we will need to add these two contributions to the final coefficient.

Let us start with $K_1$. From (2.8), we find

$$\Delta_1 = (1-2z)^2 s_{14}^2, \qquad P_{1,1} = (1-2z)k_1, \qquad P_{1,2} = -(1-2z)k_4.$$

For the spinors, we choose

$$|P_{1,1}\rangle = |1\rangle\,, \quad |P_{1,1}] = (1-2z)|1], \quad |P_{1,2}\rangle = |4\rangle\,, \quad |P_{1,2}] = -(1-2z)|4].$$

Since $n=0$, using (2.9) we get

$$-\frac{(1-2z)}{2\langle 4\ 1\rangle}\frac{d}{d\tau}\left(\frac{z^2\langle 4\ 1\rangle^2[4\ 1]^2\langle 1\ 4\rangle^2}{\langle k_4-\tau k_1|Q_2Q_1|k_4-\tau k_1\rangle}+\frac{z^2\tau^4\langle 4\ 1\rangle^2[4\ 1]^2\langle 1\ 4\rangle^2}{\langle k_1-\tau k_4|Q_2Q_1|k_1-\tau k_4\rangle}\right) = -\frac{s_{12}s_{41}^2}{2\langle 4|3|1]^2}$$

With the prefactor included, we have

$$-\frac{u[2\ 3]}{2\langle 2\ 3\rangle}C[Q_1,K_{41}] = \frac{us_{41}[2\ 3]^2[3\ 4]^2}{4s_{12}[3\ 1]^2}$$

For the triangle with $K_2$ we have

$$\Delta_2 = (1-2z)^2 s_{23}^2, \quad P_{2,1} = -(1-2z)k_2, \quad P_{2,2} = (1-2z)k_3.$$

Similar calculations give

$$-\frac{u[2\ 3]}{2\langle 2\ 3\rangle}C[Q_2,K_{41}] = -\frac{u[2\ 3]^2[4\ 3]^2}{4s_{23}[3\ 1]^2}\left(\frac{s_{13}^2}{s_{12}}-2s_{13}-s_{12}\right).$$

Adding these two contributions together, we find that the triangle coefficient is

$$-\frac{u[2\ 3]^2[4\ 3]^2s_{12}(s_{41}-s_{13})}{2s_{23}s_{12}[3\ 1]^2}. \tag{4.3}$$

**Bubble:** The formula (2.10) reduces to

$$C[K] = \frac{[\eta|RK|\eta]^2}{[\eta|Q_1K|\eta][\eta|Q_2K|\eta]} - \sum_{r=1}^{2}\left\{\frac{K^2}{\sqrt{\Delta_r}}\frac{\langle P_{r,1}|\eta|P_{r,1}]}{\langle P_{r,1}|K|P_{r,1}]}\frac{\langle P_{r,1}|R|P_{r,2}]^2}{\langle P_{r,1}|\eta|P_{r,2}]\,\langle P_{r,1}|Q_2|P_{r,2}]} - \{P_{r,1}\leftrightarrow P_{r,2}\}\right\}.$$

Taking $\eta = k_4$ it is easy to see that both the first term and the $r=1$ terms are zero, so we are left with the $r=2$ terms only. The result is

$$-\frac{[4\ 3]^2}{[3\ 1]^2}\frac{s_{13}}{s_{41}}\left(1-\frac{s_{13}}{s_{12}}\right).$$

With the prefactor included, we find that the bubble coefficient is

$$-\frac{u[2\ 3]}{2\langle 2\ 3\rangle}C[K] = \frac{us_{13}(s_{12}-s_{13})[2\ 3]^2[3\ 4]^2}{2s_{41}^2s_{12}[1\ 3]^2}. \tag{4.4}$$

**4.3** $(1^-, 2^+, 3^-, 4^+)$

The integrand for the cut $K_{41}$ is given by

$$\frac{2\left\langle 1|\ell|4\right]^2 \left\langle 3|\ell|2\right]^2}{s_{41}^2((\ell-k_4)^2-\mu^2)((\ell+k_3)^2-\mu^2)}$$

For this case we have $m=4$, $k=2$, so $n=m-k=2$, $K_1=k_4, K_2=-k_3$, thus

$$R_1 = R_2 = -(1-2z)\lambda_1\widetilde{\lambda}_4, \qquad R_3 = R_4 = -(1-2z)\lambda_3\widetilde{\lambda}_2.$$
$$Q_1 = -zk_1 - (1-z)k_4, \qquad Q_2 = (1-z)k_3 + zk_2.$$

**Box:** By now it is straightforward to find that the coefficient is

$$\frac{2}{s_{41}^2}C[Q_1,Q_2,K] = \frac{\langle 1\ 3\rangle^2 s_{41}^2(8s_{12}^2+8s_{12}s_{13}u+s_{13}^2u^2)}{8\langle 2\ 4\rangle^2 s_{13}^2}. \tag{4.5}$$

**Triangle:** For the first triangle, with $K_1$, we have

$$C[Q_1,K] = \frac{1}{12(1-2z)^3\langle 1\ 4\rangle^3}\frac{d^3}{d\tau^3}\left(\frac{\langle k_1-\tau k_4|R_1Q_1|k_1-\tau k_4\rangle^2\langle k_1-\tau k_4|R_3Q_1|k_1-\tau k_4\rangle^2}{\langle k_1-\tau k_4|Q_2Q_1|k_1-\tau k_4\rangle} + \{k_1\leftrightarrow k_4\}\right).$$

In this case, the factor $\langle k_1-\tau k_4|R_1Q_1|k_1-\tau k_4\rangle$ is proportional to $\tau^2$, so the contribution of first term is zero and we have

$$\frac{2}{s_{41}^2}C[Q_1,K] = -\frac{\langle 1\ 3\rangle^2 s_{13}}{2\langle 2\ 4\rangle^2}\frac{(1+\widetilde{A})^2(2+\widetilde{A}u)}{\widetilde{A}^3}, \quad \widetilde{A}=\frac{s_{13}}{s_{12}}$$

By symmetry, for the triangle with momentum $K_2$ we just need to do exchange the labels $1\leftrightarrow 2, 3\leftrightarrow 4$, so the final coefficient is just twice what is written above, namely

$$-\frac{\langle 1\ 3\rangle^2 s_{13}}{\langle 2\ 4\rangle^2}\frac{(1+\widetilde{A})^2(2+\widetilde{A}u)}{\widetilde{A}^3}, \quad \widetilde{A}=\frac{s_{13}}{s_{12}}. \tag{4.6}$$

**Bubble:** To present this example analytically, we choose $\eta=k_4$ rather than a more generic value. However, we do run into a problem then, because of an accidental degeneracy of poles. This problem could have been avoided by choosing a generic $\eta$, but for analytic purposes, we considered this case separately, and it is presented in Appendix B.3.1. The formula (B.28) for the bubble coefficient now takes modified input, as given in (B.25), (B.26) and (B.27).

First, consider the term (B.25). Since $n=2$, there is a third derivative in $\tau$. Since the factor $[4|(K-\tau\widetilde{\eta})R_1(K+sk_4)(K-\tau\widetilde{\eta})|4]$ is proportional to $\tau^2$, this contribution vanishes. Similarly, (B.26) will be zero. The only remaining part is the $r=2$ case, (B.27), with

$$P_1 = -(1-2z)k_2, \quad P_2 = (1-2z)K_3, \quad \Delta_{r=2} = (1-2z)s_{41}^2.$$

Let us discuss $\mathcal{B}_{2,t}^{(2;a;2)}(s)$ first. Notice that

$$\langle 3-\tau 2|R_3(K+s\eta)|3-\tau 2\rangle = (1-2z)\tau\,\langle 2\ 3\rangle\, s_{12}\overline{B}(-s+\frac{\tau}{\overline{B}}(1+(1+s)\widetilde{A})).$$

We see that to get a nonzero value of $\mathcal{B}_{2,t}^{(2;a;2)}(s)$ we must have $a=2$, and more specifically,

$$\begin{aligned}
\mathcal{B}_{2,t}^{(2;a;2)}(s) &= 0, \quad a=0,1\\
\mathcal{B}_{2,t}^{(2;a=2;1)}(s) &= -\frac{(1-2z)[4\ 1]^2\overline{B}^4\,\langle 1\ 2\rangle^4}{s_{41}^3\,\langle 2\ 3\rangle^2}\frac{(-s_{12})^{t+1}}{(t+1)s_{41}^{t+1}}\frac{z^2s^2}{(1-2z)-zs)}
\end{aligned}$$

Because the factor $s^2$ appears in $\mathcal{B}_{2,t}^{(2;a=2;1)}(s)$, there is a nonzero contribution only when we take $q=2$ in the derivative with respect to $s$. But then $a-q=0$, and since $\mathcal{B}_{2,t}^{(2;a-q;2)}(s)$ is only nonzero for $a-q=2$, the contribution from this part is zero.

For $\mathcal{B}_{2,t}^{(2;a;1)}$ we have

$$\mathcal{B}_{2,t}^{(2;a;1)}(s) = \frac{\langle 1\ 3\rangle^2}{\langle 2\ 4\rangle^2\,\widetilde{A}^2 s_{41}}\frac{(-1)^a(1-2z)^{3-a}\widetilde{A}^{t+1}}{a!(t+1)(1+\widetilde{A})^{a+t+1}}\frac{d^a}{d\widetilde{\tau}^a}\left(\frac{(1-\widetilde{\tau})^{t+1}(1-z+z\widetilde{A}\widetilde{\tau})^a(s\widetilde{A}\widetilde{\tau}-1-(1+s)\widetilde{A})^2}{(1-\widetilde{\tau})^{4-a}((1-2z)-zs)}\right),$$

where $\widetilde{\tau}=\tau/B$, $B=\langle 2\ 4\rangle/\langle 3\ 4\rangle$, and $\widetilde{A}=s_{13}/s_{12}$. Summing up $(t,a)=(2,0),(1,1),(0,2)$ with $s=0$, $(t,a)=(1,0),(0,1)$ with the first derivative of $s$, and $(t,a)=(0,0)$ with the second derivative of $s$, we finally find that the coefficient is

$$\frac{2}{s_{41}^2}C[K] = -\frac{2\,\langle 1\ 3\rangle^2\, s_{13}}{\langle 2\ 4\rangle^2\, s_{41}}\frac{(12+3\widetilde{A}(6+u)+\widetilde{A}^2(4+5u))}{12\widetilde{A}^2}, \qquad \widetilde{A}=\frac{s_{13}}{s_{12}}. \tag{4.7}$$

## 5. Discussion

Since the formalism described here is based on unitarity cuts of the amplitude, it shares with other unitarity-based approaches[11] the property that the input required is simply a collection of tree-level amplitudes. These are manifestly gauge invariant and can take quite compact forms. By dealing with different cuts separately, we can attack the problem in stages.

Furthermore, our formulas separate and identify the coefficients of individual master integrals. A single unitarity cut yields, directly and separately, the coefficients of all the master integrals with the same cut propagators. Any single coefficient can be targeted individually, without the need to first compute any others or additional spurious terms.

[11] For a review, see Section 4 of [41].

## 5.1 Comparison with other approaches

The reduction algorithm of Ossola, Papadopoulos and Pittau (OPP) [42, 43, 44] produces coefficients through algebraic operations at the integrand level, through recursive solution of a set of algebraic equations. In fact, our formulas given here are the results of solving algebraic equations in a different style. In the OPP method, several points of phase space are used, while in our method, we differentiate at a single point of phase space. The derivative operator can be interpreted as an algebraic procedure as applied to rational functions at a single point.

Just like our result, coefficients from OPP method can be fed into the $d$-dimensional unitarity program as described in [23, 24]. Alternatively, the algorithm may be interpreted numerically, and in fact such an implementation has now been given [45] (see also the procedure of [46]). We believe that the formulas of the present paper are also well suited for numerical programming, and this will be the subject of forthcoming work.

One final note on comparison to the OPP method is that our formulas are valid for arbitrary values of $n$, in particular for $n > 2$. Such an extension has been mentioned within the OPP method, although details have not been worked out.

An approach that is closer in spirit to ours was given by Forde in [19]. There, coefficients for boxes, triangles and bubbles are given within the spinor formalism. The foundation there consists of generalized unitarity cuts, namely quadruple cuts for boxes, triple cuts for triangles, and ordinary double cuts for bubbles. The motivation was to capitalize on the efficiency of quadruple cuts for box coefficients, and also to be able to target specific coefficients. Forde's final formulas resemble ours in that they are based on data from tree amplitudes and given in terms of a coefficient in a series expansion of one variable for triangles, and two variables for bubbles. The formula for a bubble coefficient, however, depends on tree amplitude input for all possible triple cuts, while ours comes directly from the ordinary double cut (though still depending on all possible momenta from a hypothetical third cut). If the aim is to assemble an amplitude in its entirety, then of course the complete tree-level amplitude input will be available anyway.

In [15], we discussed the application of quadruple cuts to box and pentagon integrals in $d$ dimensions. A $d$-dimensional analysis of triple cuts for triangle integrals has been given in [18]. Let us now briefly examine the triple cut in the context of the present paper in order to make contact with the result of [19].

With three delta functions in $d$ dimensions, we first use two of them to set up the four-dimensional spinor integrand, as explained in Appendix A. After integrating over the variable $t$, defined in (A.2), we arrive at the integral

$$\int \langle \ell\ d\ell \rangle\, [\ell\ d\ell]\ G(\ell)\ \delta\left( \frac{K^2\, \langle \ell | Q | \ell ]}{\langle \ell | K | \ell ]} \right)$$

Now we can use momenta $Q, K$ to construct two null momenta as in (2.8) and expand our spinor variables in the basis of their spinor components as follows:

$$|\ell\rangle = |P_1\rangle + z\,|P_2\rangle\,, \qquad |\ell] = |P_1] + \overline{z}|P_2]$$

Here $z$ is a complex number and $\overline{z}$ is its conjugate. With this substitution, we get

$$\int dz\; d\overline{z}\; (2P_1 \cdot P_2)\; G(z,\overline{z})\; \delta\left(K^2 \frac{\langle P_1|Q|P_1] + z\overline{z}\,\langle P_2|Q|P_2]}{\langle P_1|Q|P_1] + z\overline{z}\,\langle P_2|Q|P_2]}\right). \tag{5.1}$$

Now we can change to polar coordinates so that $z = re^{i\theta}$ and $dzd\overline{z} = rdrd\theta$. Furthermore, if we now define the new variable $t = e^{i\theta}$, we have

$$dzd\overline{z} = rdr \times \frac{-idt}{t}.$$

The delta function depends only on $r$, so we can use it to integrate over $r$. Then we are left with $t$ integration only. From here, for example, it is easy to see the vanishing condition given in eq. (4.20) of [19].

Furthermore, for box integrals, we have an extra propagator, so the general form of the integrand is $1/(a + tb + t^{-1}c)$. Only polynomial terms correspond to the triangle contribution.

The parametrization we have used here is not exactly the one used by Forde, but the central idea is the same and $t$ is the angle variable for both triple cuts and double cuts.

### 5.2 Prospects

The most obvious and immediate application of our results will be to the computation of complete one-loop amplitudes, as in the example of Section 4, where the $u$-dependent expressions for coefficients are fed into the reduction formulas of [23, 24] to give the final $\epsilon$-dependent coefficients. The four-momenta of the external particles may be numerical at every step. The reduction formulas currently require analytic expressions in $u$.

Our formulas may also be specialized to the cut-constructible part of the amplitude, as in the example of Section 3, simply by setting $u \to 0$ at the end of the calculation and interpreting the formulas as exact coefficients of 4-dimensional master integrals.

Since several methods proposed in the literature are specialized for computing either cut-constructible or rational components of an amplitude, it would also be very interesting to specialize our formulas to isolate the rational part of one-loop amplitudes. This could be done by studying the $\epsilon$-dependence of the reduction formulas together with the $u$-dependence in the coefficients, in order to focus precisely on the $\epsilon^0$ term in the final $\epsilon$-expansion of the amplitude. We will return to this point in a future publication.

Finally, as we remarked in the introduction, our formulas apply to amplitudes with massive or massless propagators. In order to arrive at complete amplitudes in the massive case, the master integrals should be evaluated explicitly,[12] and our results will need to be supplemented with the contributions of tadpole and massless bubble integrals.

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## A. Setting up the cut integral

In this appendix we briefly review the first steps in spinor integration, in the context of $d$-dimensional unitarity, leading from equation (1.1) to equation (1.5). For a fuller discussion of this technique, see [24]. Within the four-dimensional helicity scheme, we apply (1.2) and (1.3). In the integrand, $p$ is replaced by $\widetilde{\ell}$, and the measure is transformed as follows:

$$\int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} = \int \frac{d^4\widetilde{\ell}}{(2\pi)^4}\frac{(4\pi)^\epsilon}{\Gamma(-\epsilon)}\int d\mu^2\ (\mu^2)^{-1-\epsilon},$$

We now drop the factor $\frac{(4\pi)^\epsilon}{(2\pi)^4\Gamma(-\epsilon)}\left(\frac{K^2}{4}\right)^{-\epsilon}$, which is universal and common to cuts of amplitudes and master integrals. Following [23, 24], we further decompose the 4-dimensional momentum into a null component and a component proportional to the cut momentum $K$.

$$\widetilde{\ell} = \ell + zK, \qquad \ell^2 = 0, \qquad \Longrightarrow \int d^4\widetilde{\ell} = \int dz\ d^4\ell\ \delta^+(\ell^2)(2\ell\cdot K).$$

While changing the variable $\mu$ to $u$ with (1.3), we note that the kinematics of the unitarity cut constrain the integration domain to be $u \in [0,1]$. Our cut integral (1.1) can now been rewritten as the following expression.

$$\int_0^1 du\ u^{-1-\epsilon}\int dz\ (1-2z)\delta(z(1-z)-\frac{u}{4})$$
$$\int d^4\ell\ \delta^+(\ell^2)\delta((1-2z)K^2-2\ell\cdot K)\frac{\prod_{i=1}^M(-2P_i\cdot(\ell+zK))}{\prod_{j=1}^N(K_j^2-z(2K_j\cdot K)-2\ell\cdot K_j)}$$

[12] For a uniform mass, this has been done in [22].

Notice that the $z$-integral can now be done with the first delta function. In fact, the kinematics of the unitarity cut require us to choose exactly one solution for $z$. If we take $K > 0$, then

$$z = \frac{1-\sqrt{1-u}}{2}, \quad \text{or equivalently,} \quad 1-2z = \sqrt{1-u}. \tag{A.1}$$

Now we change to spinor variables with [47]

$$\ell = t\lambda\widetilde{\lambda}, \tag{A.2}$$

where $t$ takes nonnegative real values, and $\lambda$ and $\widetilde{\lambda}$ are homogeneous spinors. The measure transforms as

$$\int d^4\ell\ \delta^{(+)}(\ell^2)\ (\bullet) = \int_0^\infty dt\ t \int_{\tilde{\lambda}=\bar{\lambda}} \langle\lambda\ d\lambda\rangle\,[\tilde{\lambda}\ d\tilde{\lambda}](\bullet).$$

The domain of integration of $t$ is again consistent with the kinematic region of the unitarity cut. From here on, we use $|\ell\rangle$ and $|\ell]$ interchangeably with $\lambda$ and $\tilde{\lambda}$. We have now arrived at the following expression:

$$C = \int du\ u^{-1-\epsilon} \int \langle\ell\ d\ell\rangle\,[\ell\ d\ell] \int t\ dt\ \delta((1-2z)K^2 + t\,\langle\ell|K|\ell]) \frac{\prod_{i=1}^{M}(-z(2K\cdot P_i) + t\,\langle\ell|P_i|\ell])}{\prod_{j=1}^{N}(K_j^2 - z(2K_j\cdot K) + t\,\langle\ell|K_j|\ell])}$$

Finally, we use the remaining delta function to perform the integral over the variable $t$. With the substitution (A.1), the result is equation (1.5).

## B. Derivation of the formulas for coefficients

In this appendix we outline the derivation of the main results of this paper, which are the formulas (2.7), (2.9) and (2.10). Our technique is the type of spinor integration carried out in [7, 8, 15], but we stress that understanding these techniques is unnecessary for applying the results. Indeed, equivalent formulas have already appeared in [15].[13] The difference is that our starting point is now the raw unitarity cut integral, before converting the loop momentum to spinor variables. In the final formulas, we now explicitly evaluate the residue at multiple poles. Additionally, the present versions of the formulas feature substantial simplification of the bubble coefficients.

Our foundation here is the framework laid out in [15] and its references. Let us briefly recall the key ideas. The general integrand given as the starting point in [15] is [14]

$$I_{term} = \frac{G(\lambda)\prod_{j=1}^{n+k}[a_j\ \ell]}{\langle\ell|K|\ell]^{n+2}\prod_{p=1}^{k}\langle\ell|Q_p|\ell]}. \tag{B.1}$$

[13] The formulas for coefficients given in [15] differ from the ones given here by a factor of $\sqrt{1-u}$. This comes from the convention of our starting point (1.1) or equivalently (1.5), where this factor appears explicitly.

[14] We have redefined the index $n$ for consistency.

Comparing with the expression (1.5), we see that we will take $G(\lambda)$ to be constant and $[a_j| = \langle \ell|R_j|$. The idea of spinor integration is to rewrite the integral so that we can carry it out with the residue theorem. The next step, therefore, is to isolate poles by splitting the denominator factors with spinor identities such as

$$\frac{[a\ \ell]}{\langle \ell|Q_1|\ell]\ \langle \ell|Q_2|\ell]} = \frac{[a|Q_1|\ell\rangle}{\langle \ell|Q_2Q_1|\ell\rangle\ \langle \ell|Q_1|\ell]} + \frac{[a|Q_2|\ell\rangle}{\langle \ell|Q_1Q_2|\ell\rangle\ \langle \ell|Q_2|\ell]}. \tag{B.2}$$

This procedure is applied to the amplitude on one hand and the master integrals on the other. By matching functional forms, we extract the coefficients.

### B.1 Box

The formula (2.7) for a box coefficient is trivially related to the one given in [15]. We only need to observe that now that we take $G(\lambda)$ to be constant and $[a_j| = \langle \ell|R_j|$ in (B.1), the poles from the factors $\langle \ell|Q_sQ_r|\ell\rangle$ are inserted into $[a_j|$ as well when we evaluate the residue.

### B.2 Triangle

For a triangle associated to momenta $K_s, K$, the coefficient was found in [15] to be the difference of the two residues from the poles in $\langle \ell|Q_sK|\ell\rangle^{n+2}$ of the following function:[15]

$$\frac{(-1)^n (K^2)^{1+n}\sqrt{\Delta_s}}{2} \frac{\prod_{j=1}^{k+n} \langle \ell|R_jQ_s|\ell\rangle}{\langle \ell|Q_sK|\ell\rangle^{n+2} \prod_{t=1,t\neq s}^{k} \langle \ell|Q_tQ_s|\ell\rangle}\ . \tag{B.3}$$

The quantities $\Delta_s, P_{s,1}, P_{s,2}$ were defined in (2.8) specifically to deal with the factor $\langle \ell|Q_sK|\ell\rangle$ by identifying the poles explicitly. With those definitions, we find an identity that separates the two poles:

$$\langle \ell|Q_sK|\ell\rangle = \langle \ell\ P_{s,1}\rangle\ \langle \ell\ P_{s,2}\rangle \frac{K^2[P_{s,1}\ P_{s,2}]}{\sqrt{\Delta_s}} \quad . \tag{B.4}$$

Now consider the residue from a multiple pole, in an expression of the form

$$\frac{1}{\langle \ell\ \eta\rangle^n} \frac{N(|\ell\rangle, |\ell])}{D(|\ell\rangle, |\ell])}.$$

We can start by substituting $|\ell] = |\eta]$, so we are dealing with the holomorphic function

$$\frac{1}{\langle \ell\ \eta\rangle^n} \frac{N(|\ell\rangle, |\eta])}{D(|\ell\rangle, |\eta])}.$$

[15] Here we have again redefined $n$ and made substitutions for $G(\lambda)$ and $[a_j|$.

For an arbitrary auxiliary spinor $\zeta$, we have the following identity.

$$\frac{1}{\langle \ell\ (\eta - \tau\zeta)\rangle^n} = \frac{d^{n-1}}{d\tau^{n-1}}\left(\frac{1}{(n-1)!\,\langle \ell\ \zeta\rangle^{n-1}}\frac{1}{\langle \ell\ (\eta-\tau\zeta)\rangle}\right)\Bigg|_{\tau\to 0} \tag{B.5}$$

Thus we find

$$\frac{1}{\langle \ell\ (\eta - \tau\zeta)\rangle^n} = \frac{d^{n-1}}{d\tau^{n-1}}\left(\frac{1}{(n-1)!\,\langle \ell\ \zeta\rangle^{n-1}}\frac{1}{\langle \ell\ (\eta-\tau\zeta)\rangle}\frac{N(|\ell\rangle\,,|\eta])}{D(|\ell\rangle\,,|\eta])}\right)\Bigg|_{\tau\to 0}\ . \tag{B.6}$$

Now we extract the residue at the single pole $\langle \ell\ (\eta-\tau\zeta)\rangle$ before taking the derivative. We find

$$\frac{d^{n-1}}{d\tau^{n-1}}\left(\frac{1}{(n-1)!\,\langle \eta\ \zeta\rangle^{n-1}}\frac{N(|\eta-\tau\zeta\rangle\,,|\eta])}{D(|\eta-\tau\zeta\rangle\,,|\eta])}\right)\Bigg|_{\tau\to 0}\ . \tag{B.7}$$

To obtain the residues from the factor $\langle \ell|QK|\ell\rangle^n$, we use equation (B.4) to rewrite it in terms of two multiple poles.[16] Then we apply (B.7) to compute the two residues as follows:

$$R_1 = \frac{d^{n-1}}{d\tau_1^{n-1}}\left(\frac{1}{(n-1)!\,\langle P_1\ \zeta_1\rangle^{n-1}}\frac{(K^2)^n N(|P_1-\tau_1\zeta_1\rangle\,,|P_1])}{(\sqrt{\Delta})^n[P_1\ P_2]^n\,\langle P_1-\tau_1\zeta_1,P_2\rangle^n\,D(|P_1-\tau_1\zeta_1\rangle\,,|P_1])}\right)\Bigg|_{\tau\to 0},$$
$$R_2 = \frac{d^{n-1}}{d\tau_2^{n-1}}\left(\frac{1}{(n-1)!\,\langle P_2\ \zeta_2\rangle^{n-1}}\frac{(K^2)^n N(|P_2-\tau_2\zeta_2\rangle\,,|P_2])}{(\sqrt{\Delta})^n[P_1\ P_2]^n\,\langle P_2-\tau_2\zeta_2,P_1\rangle^n\,D(|P_2-\tau_2\zeta_2\rangle\,,|P_2])}\right)\Bigg|_{\tau\to 0}.$$

where we can choose different auxiliary spinors $\zeta_1,\zeta_2$ for the two poles. To simplify further, we choose $|\zeta_1\rangle = |P_2\rangle$ and $|\zeta_2\rangle = |P_1\rangle$ and use the identity $[P_1\ P_2]\,\langle P_1\ P_2\rangle = -\Delta/K^2$. Finally we find

$$R_1 = \frac{(-1)^n}{(\sqrt{\Delta})^n}\frac{1}{(n-1)!\,\langle P_1\ P_2\rangle^{n-1}}\frac{d^{n-1}}{d\tau_1^{n-1}}\left(\frac{N(|P_1-\tau_1P_2\rangle\,,|P_1])}{D(|P_1-\tau_1P_2\rangle\,,|P_1])}\right)\Bigg|_{\tau\to 0}, \tag{B.8}$$

$$R_2 = -\frac{(-1)^n}{(\sqrt{\Delta})^n}\frac{1}{(n-1)!\,\langle P_1\ P_2\rangle^{n-1}}\frac{d^{n-1}}{d\tau_2^{n-1}}\left(\frac{N(|P_2-\tau_2P_1\rangle\,,|P_2])}{D(|P_2-\tau_2P_1\rangle\,,|P_2])}\right)\Bigg|_{\tau\to 0}. \tag{B.9}$$

Using (B.8) and (B.9) with our original expression (B.3), we get the formula (2.9) for the triangle coefficient. That formula may look as though it is not completely explicit because we still need to perform a differentiation. But this is easily done in a symbolic manipulation program. We do offer explicit formulas in Appendix C but do not expect those to be more useful.

Recall that the final result must be a rational function, so the square roots from $\sqrt{\Delta_s}$ should eventually combine into polynomial expressions.

[16]Throughout the rest of this derivation, we drop the subscript $s$ to avoid cluttering the formulas.

### B.3 Bubble

Our derivation here parallels the one in [15], but the splitting identities are more systematic and the final formula is now written explicitly.

First, we would like to split the denominator factors in (B.1) using the following generalization of (B.2):

$$\frac{\prod_{j=1}^{k-1}[a_j\ \ell]}{\prod_{i=1}^{k}\langle\ell|Q_i|\ell]} = \sum_{i=1}^{k}\frac{1}{\langle\ell|Q_i|\ell]}\frac{\prod_{j=1}^{k-1}[a_j|Q_i|\ell\rangle}{\prod_{j=1,j\neq i}^{k}\langle\ell|Q_jQ_i|\ell\rangle} \tag{B.10}$$

This formula is applicable when and only when all $Q_i$ and $K$ are different. To use it, we deform (B.1) by introducing small independent parameters $s_i$, $i=1,...,n+1$ and a *real* null vector $\eta$.

$$\frac{G(\lambda)\prod_{j=1}^{n+k}[a_j\ \ell]}{\langle\ell|K|\ell]\prod_{i=1}^{n+1}\langle\ell|K+s_i\eta|\ell]\prod_{p=1}^{k}\langle\ell|Q_p|\ell]} \tag{B.11}$$

The final result will be recovered by taking the limit $s_i\to 0$.

Now we can apply (B.10) to (B.11) to find the following expression:

$$\sum_{i=1}^{n+1}\frac{1}{\langle\ell|K|\ell]\,\langle\ell|K+s_i\eta|\ell]}\frac{G(\lambda)\prod_{j=1}^{n+k}[a_j|K+s_i\eta|\ell\rangle}{\prod_{q=1,q\neq i}^{n+1}\langle\ell|(K+s_q\eta)(K+s_i\eta)|\ell\rangle\prod_{p=1}^{k}\langle\ell|Q_p(K+s_i\eta)|\ell\rangle} \tag{B.12}$$

$$+\sum_{i=1}^{k}\frac{1}{\langle\ell|K|\ell]\,\langle\ell|Q_i|\ell]}\frac{(G(\lambda)\prod_{j=1}^{n+k}[a_j|Q_i|\ell\rangle}{\prod_{q=1}^{n+1}\langle\ell|(K+s_q\eta)Q_i|\ell\rangle\prod_{r=1,r\neq i}^{k}\langle\ell|Q_rQ_i|\ell\rangle} \tag{B.13}$$

We can see that the $s_i\to 0$ limit is smooth in the second line, resulting in terms of the form $F_i(\lambda)/(\langle\ell|K|\ell]\,\langle\ell|Q_i|\ell])$. We know from [8] that these terms yield pure logarithms, in this case for the triangle integrals associated with momenta $K$ and $K_i$. So we restrict our attention to the first line, (B.12). Rewrite it as

$$\sum_{i=1}^{n+1}\frac{1}{\langle\ell|K|\ell]\,\langle\ell|K+s_i\eta|\ell]}\frac{G(\lambda)\prod_{j=1}^{n+k}[a_j|K+s_i\eta|\ell\rangle}{\langle\ell|K\eta|\ell\rangle^n\prod_{q=1,q\neq i}^{n+1}(s_i-s_q)\prod_{p=1}^{k}\langle\ell|Q_p(K+s_i\eta)|\ell\rangle}$$

Now we can must take the limit $s_i\to 0$ carefully. We find that the bubble coefficient is

$$\sum_{q=0}^{n}\frac{(-1)^q}{q!}\frac{d^qB_{n,n-q}(s)}{ds^q}\bigg|_{s=0}, \tag{B.14}$$

where we have defined the function

$$B_{n,t}(s)\equiv\frac{\langle\ell|\eta|\ell]^t}{\langle\ell|K|\ell]^{2+t}}\frac{G(\lambda)\prod_{j=1}^{n+k}[a_j|K+s\eta|\ell\rangle}{\langle\ell|\eta K|\ell\rangle^n\prod_{p=1}^{k}\langle\ell|Q_p(K+s\eta)|\ell\rangle}\ . \tag{B.15}$$

The fact that (B.14) represents the bubble coefficient can be proved by induction. The case $n=0$ is trivial. Assume that it is true for $n$, and let us now introduce a single parameter $\widetilde{s}$ to rewrite (B.11) as

$$\frac{G(\lambda)\prod_{j=1}^{n+k+1}[a_j\ \ell]}{\langle\ell|K|\ell]^{n+2}\langle\ell|K+\widetilde{s}\eta|\ell]\prod_{p=1}^{k}\langle\ell|Q_p|\ell]}. \tag{B.16}$$

We now treat the factor $\langle\ell|K+\widetilde{s}\eta|\ell]$ on the same footing as $\langle\ell|Q_p|\ell]$ and apply the result for $n$. The bubble contribution can be expressed as a sum of two terms $I_1$ and $I_2$. The first term is

$$\begin{aligned}
I_1 &= \frac{1}{\langle\ell|K|\ell]\,\langle\ell|K+\widetilde{s}\eta|\ell]}\frac{G(\lambda)\prod_{j=1}^{n+k+1}[a_j|K+\widetilde{s}\eta|\ell\rangle}{\langle\ell|K(K+\widetilde{s}\eta)|\ell\rangle^{n+1}\prod_{p=1}^{k}\langle\ell|Q_p(K+\widetilde{s}\eta)|\ell\rangle} \\
&= \frac{(-1)^{n+1}}{\widetilde{s}^{n+1}}\left(\sum_t(-1)^t\widetilde{s}^t\frac{\langle\ell|\eta|\ell]^t}{\langle\ell|K|\ell]^{t+2}}\right)\frac{G(\lambda)\prod_{j=1}^{n+k+1}[a_j|K+\widetilde{s}\eta|\ell\rangle}{\langle\ell|\eta K|\ell\rangle^{n+1}\prod_{p=1}^{k}\langle\ell|Q_p(K+\widetilde{s}\eta)|\ell\rangle}
\end{aligned}$$

After taking the $\widetilde{s}\to 0$ limit, we have

$$I_1 = \sum_{a=0}^{n+1}\sum_{t=0}^{n+1-a}\frac{(-1)^{n+1+t}}{\widetilde{s}^{n+1-t-a}a!}\frac{d^a B_{n+1,t}(\widetilde{s}=0)}{d\widetilde{s}^a}, \tag{B.17}$$

where $B_{n,t}(s)$ is defined by (B.15).

The second contribution is

$$I_2 = \sum_{q=0}^{n}\frac{(-1)^q}{q!}\frac{d^q\widetilde{B}_{n,n-q}(s=0)}{ds^q},$$

where

$$\begin{aligned}
\widetilde{B}_{n,t}(s) &\equiv \frac{\langle\ell|\eta|\ell]^t}{\langle\ell|K|\ell]^{2+t}}\frac{G(\lambda)\prod_{j=1}^{n+k+1}[a_j|K+s\eta|\ell\rangle}{\langle\ell|\eta K|\ell\rangle^n\langle\ell|(K+\widetilde{s}\eta)(K+s\eta)|\ell\rangle\prod_{p=1}^{k}\langle\ell|Q_p(K+s\eta)|\ell\rangle} \\
&= \frac{1}{(\widetilde{s}-s)}B_{n+1,t}(s).
\end{aligned} \tag{B.18}$$

We must take the $s\to 0$ limit before $\widetilde{s}\to 0$, so first we substitute

$$\left.\frac{d^q\widetilde{B}_{n,n-q}(s)}{ds^q}\right|_{s=0} = \sum_{b=0}^{q}\frac{q!}{(q-b)!\widetilde{s}^{1+b}}\frac{d^{q-b}}{ds^{q-b}}B_{n+1,t}(s=0)$$

to find

$$I_2 = \sum_{q=0}^{n}\sum_{b=0}^{q}\frac{(-1)^q}{(q-b)!\widetilde{s}^{1+b}}\frac{d^{q-b}}{ds^{q-b}}B_{n+1,n-q}(s=0),$$

or equivalently,

$$I_2 = \sum_{a=0}^{n}\sum_{t=0}^{n-a} \frac{(-1)^{n-t}}{a!\widetilde{s}^{1+n-t-a}} \frac{d^a}{ds^a} B_{n+1,t}(s=0) \tag{B.19}$$

Now it is easy to see that $I_1 + I_2$ is nonzero only if $a + t = n + 1$. Therefore we can write

$$I_1 + I_2 = \sum_{a=0}^{n+1} \frac{(-1)^a}{a!} \frac{d^a}{ds^a} B_{n+1,n+1-a}(s=0), \tag{B.20}$$

and thus we have proved the formula (B.14) for $n + 1$.

Now that we have established that the bubble coefficient comes from (B.14) with the definition (B.15), we need to identify the poles and find the residues.

Rewrite the integrand (B.14) as a total derivative by using

$$[\ell\ d\ell] B_{n,t}(s) = [d\ell\ \partial_\ell] \left( \frac{G(\lambda)}{(t+1)} \frac{\langle\ell|\eta|\ell]^{t+1}}{\langle\ell|K|\ell]^{t+1}} \frac{\prod_{j=1}^{n+k} [a_j|K+s\eta|\ell\rangle}{\langle\ell|\eta K|\ell\rangle^{n+1} \prod_{p=1}^{k} \langle\ell|Q_p(K+s\eta)|\ell\rangle} \right)$$

Now let us specialize to the integrand of (1.5), so that $G(\lambda)$ is constant and $[a_j| = \langle\ell|R_j|$. Now we define [17]

$$\mathcal{B}_{n,t}(s) \equiv \frac{1}{(t+1)} \frac{\langle\ell|\eta|\ell]^{t+1}}{\langle\ell|K|\ell]^{t+1}} \frac{\prod_{j=1}^{n+k} \langle\ell|R_j(K+s\eta)|\ell\rangle}{\langle\ell|\eta K|\ell\rangle^{n+1} \prod_{p=1}^{k} \langle\ell|Q_p(K+s\eta)|\ell\rangle} \tag{B.21}$$

Here it is important that $\eta$ be completely generic, so that there are no accidental degeneracies. For an alternative approach, see Subsection B.3.1. There are three kinds of poles. The first, at $\ell = \eta$, has no residue because the numerator factor $\langle\ell|\eta|\ell]^{t+1}$ becomes zero. The second, at $|\ell\rangle = K|\eta]$, is a multiple pole of the type discussed in B.2, so we see that its residue is (2.11). The last kind of pole is from the factor $\langle\ell|Q_r(K+s\eta)|\ell\rangle$. Here we perform a series expansion in the parameter $s$, which we will ultimately set to zero. The expansion is

$$\frac{1}{\langle\ell|Q_r(K+s\eta)|\ell\rangle} = \sum_{a=0} (-s)^a \frac{\langle\ell|Q_r\eta|\ell\rangle^a}{\langle\ell|Q_r K|\ell\rangle^{a+1}} . \tag{B.22}$$

The residue is then $\mathcal{B}_{n,t}^{(r;a;1)}(s) - \mathcal{B}_{n,t}^{(r;a;2)}(s)$, with the definitions given in (2.12) and (2.13).

Combining these contributions, we find that the sum of the residues at poles of $\mathcal{B}_{n,t}(s)$ is

$$\mathcal{B}_{n,t}^{(0)}(s) + \sum_{r=1}^{k}\sum_{a=0} (-s)^a \left( \mathcal{B}_{n,t}^{(r;a;1)}(s) - \mathcal{B}_{n,t}^{(r;a;2)}(s) \right) . \tag{B.23}$$

Feeding (B.23) into (B.14) and simplifying the result gives us our final expression for the bubble coefficient, (2.10).

[17] The $B_{n,t}(s)$ is the splitting result while $\mathcal{B}_{n,t}(s)$ is after writing into total derivative, i.e.,$B_{n,t}(s) = [d\widetilde{\lambda}\ \partial_{\widetilde{\lambda}}]\mathcal{B}_{n,t}(s)$.

### B.3.1 A special choice of $\eta$

In this appendix, we describe the consequences of choosing $\eta = K_1$ in the case where $K_1^2$. This choice may be convenient for small examples worked by hand, but we emphatically recommend choosing a generic $\eta$ wherever possible.

The reason that such a special choice of $\eta$ presents a problem is the following. From (1.4), we can see that

$$\langle\ell|Q_1K|\ell\rangle = -(1-2z)\,\langle\ell|K_1K|\ell\rangle = -(1-2z)\,\langle\ell|\eta K|\ell\rangle$$

Therefore, in the expression (B.21), the poles from $\langle\ell|\eta K|\ell\rangle$ and $\langle\ell|Q_1(K+s\eta)|\ell\rangle$, will overlap, and the way we read off their residues should be modified respectively.

In this special case, it is easy to see that

$$\frac{1}{\langle\ell|Q_1(K+s\eta)|\ell\rangle} = \sum_{a=0}(-s)^a\frac{\langle\ell|Q_1\eta|\ell\rangle^a}{\langle\ell|Q_1K|\ell\rangle^{a+1}} = -\frac{1}{\langle\ell|\eta K|\ell\rangle}\frac{1}{(1-2z)-sz\frac{(2K\cdot\eta)}{K^2}}$$

Now instead of (B.21), we have

$$\mathcal{B}_{n,t}(s) \equiv -\frac{1}{(1-2z)-sz\frac{(2K\cdot\eta)}{K^2}}\frac{1}{(t+1)}\frac{\langle\ell|\eta|\ell]^{t+1}}{\langle\ell|K|\ell]^{t+1}}\frac{\prod_{j=1}^{n+k}\langle\ell|R_j(K+s\eta)|\ell\rangle}{\langle\ell|\eta K|\ell\rangle^{n+2}\prod_{p=2}^{k}\langle\ell|Q_p(K+s\eta)|\ell\rangle}. \tag{B.24}$$

Continuing this way, we find

$$\begin{aligned}\mathcal{B}^{(0)}_{n,t}(s) \equiv -\frac{d^{n+1}}{d\tau^{n+1}}\Bigg(&\frac{1}{(1-2z)-sz\frac{(2K\cdot\eta)}{K^2}}\frac{[\eta|\widetilde{\eta}K|\eta]^{-n-1}}{(t+1)(n+1)!}\left(\frac{(2\eta\cdot K)}{K^2}\right)^{t+1}\\ &\frac{\prod_{j=1}^{n+k}\langle\ell|R_j(K+s\eta)|\ell\rangle}{\langle\ell\ \eta\rangle^{n+2}\prod_{p=2}^{k}\langle\ell|Q_p(K+s\eta)|\ell\rangle}\Big|_{|\ell\rangle\to|K-\tau\widetilde{\eta}|\eta]}\Bigg)\Bigg|_{\tau\to 0},\end{aligned} \tag{B.25}$$

$$\begin{aligned}\mathcal{B}^{(r;a;1)}_{n,t}(s) \equiv\ &\frac{1}{(1-2z)-sz\frac{(2K\cdot\eta)}{K^2}}\frac{(-1)^a}{\sqrt{\Delta_r}^{a+1}a!\,\langle P_{r,1}\ P_{r,2}\rangle^a}\frac{d^a}{d\tau^a}\Bigg(\frac{1}{(t+1)}\frac{\langle\ell|\eta|\ell]^{t+1}}{\langle\ell|K|\ell]^{t+1}}\\ &\times\frac{\langle\ell|Q_r\eta|\ell\rangle^a\prod_{j=1}^{n+k}\langle\ell|R_j(K+s\eta)|\ell\rangle}{\langle\ell|\eta K|\ell\rangle^{n+2}\prod_{p=1,p\neq r}^{k}\langle\ell|Q_p(K+s\eta)|\ell\rangle}\Bigg)\Bigg|_{|\ell]=|P_{r,1}],|\ell\rangle=|P_{r,1}\rangle-\tau|P_{r,2}\rangle}\end{aligned} \tag{B.26}$$

$$\begin{aligned}\mathcal{B}^{(r;a;2)}_{n,t}(s) \equiv\ &\frac{1}{(1-2z)-sz\frac{(2K\cdot\eta)}{K^2}}\frac{(-1)^a}{\sqrt{\Delta_r}^{a+1}a!\,\langle P_{r,1}\ P_{r,2}\rangle^a}\frac{d^a}{d\tau^a}\Bigg(\frac{1}{(t+1)}\frac{\langle\ell|\eta|\ell]^{t+1}}{\langle\ell|K|\ell]^{t+1}}\\ &\times\frac{\langle\ell|Q_r\eta|\ell\rangle^a\prod_{j=1}^{n+k}\langle\ell|R_j(K+s\eta)|\ell\rangle}{\langle\ell|\eta K|\ell\rangle^{n+2}\prod_{p=1,p\neq r}^{k}\langle\ell|Q_p(K+s\eta)|\ell\rangle}\Bigg)\Bigg|_{|\ell]=|P_{r,2}],|\ell\rangle=|P_{r,2}\rangle-\tau|P_{r,1}\rangle}\end{aligned} \tag{B.27}$$

and the coefficient is given by

$$C[K]_n = (K^2)^{1+n} \sum_{q=0}^{n} \frac{(-1)^q}{q!} \frac{d^q}{ds^q} \left( \mathcal{B}^{(0)}_{n,n-q}(s) + \sum_{r=2}^{k} \sum_{a=q}^{n} \left( \mathcal{B}^{(r;a-q;1)}_{n,n-a}(s) - \mathcal{B}^{(r;a-q;2)}_{n,n-a}(s) \right) \right) \Bigg|_{s=0} . \tag{B.28}$$

where the definitions of the functions $\mathcal{B}$ must be taken from (B.25), (B.26) and (B.27).

## C. Closed forms for triangle coefficients

We have given the general expression for coefficients of triangles as a formula involving a multiple derivative (2.9). We can also carry out the differentiation explicitly. For practical purposes, we need to consider only cases with $n \leq 2$.

When $n \leq -2$, the contribution is simply zero.

When $n = -1$, there is no derivative, so the result is just

$$C[Q_s, K]_{n=-1} = \frac{1}{2} \left( \frac{\prod_{j=1}^{k-1} \langle P_{s,1}|R_j|P_{s,2}]}{\prod_{t=1,t\neq s}^{k} \langle P_{s,1}|Q_t|P_{s,2}]} \right) \tag{C.1}$$

When $n = 0$ it is given by

$$\begin{aligned} C[Q_s, K]_{n=0} = \frac{K^2}{2\Delta_s} \Bigg\{ & \frac{\prod_{j=1}^{k} \langle P_{s,1}|R_j|P_{s,2}]}{\prod_{t=1,t\neq s}^{k} \langle P_{s,1}|Q_t|P_{s,2}]} \left( \sum_{j=1}^{k} \frac{(2Q_s \cdot K)(2R_j \cdot Q_s) - 2Q_s^2(2R_j \cdot K)}{\langle P_{s,1}|R_j|P_{s,2}]} \right. \\ & \left. - \sum_{t=1,t\neq s}^{k} \frac{(2Q_s \cdot K)(2Q_t \cdot Q_s) - 2Q_s^2(2Q_t \cdot K)}{\langle P_{s,1}|Q_t|P_{s,2}]} \right) + \{P_{s,1} \leftrightarrow P_{s,2}\} \Bigg\} \end{aligned} \tag{C.2}$$

When $n = 1$ it will be

$$\begin{aligned} C[Q_s, K]_{n=1} = \frac{(K^2)^2}{4\Delta_s^2} \Bigg\{ & \frac{\prod_{j=1}^{k+1} \langle P_{s,1}|R_j|P_{s,2}]}{\prod_{t=1,t\neq s}^{k} \langle P_{s,1}|Q_t|P_{s,2}]} \Bigg[ \left( \sum_{j=1}^{k+1} \frac{(2Q_s \cdot K)(2R_j \cdot Q_s) - 2Q_s^2(2R_j \cdot K)}{\langle P_{s,1}|R_j|P_{s,2}]} \right. \\ & \left. - \sum_{t=1,t\neq s}^{k} \frac{(2Q_s \cdot K)(2Q_t \cdot Q_s) - 2Q_s^2(2Q_t \cdot K)}{\langle P_{s,1}|Q_t|P_{s,2}]} \right)^2 \\ & + \sum_{j=1}^{k+1} \frac{-[(2Q_s \cdot K)(2R_j \cdot Q_s) - 2Q_s^2(2R_j \cdot K)]^2 + 2Q_s^2 K^2 \langle P_{s,1}|R_j|P_{s,2}] \langle P_{s,2}|R_j|P_{s,1}]}{\langle P_{s,1}|R_j|P_{s,2}]^2} \\ & - \sum_{t=1,t\neq s}^{k} \frac{-[(2Q_s \cdot K)(2Q_t \cdot Q_s) - 2Q_s^2(2Q_t \cdot K)]^2 + 2Q_s^2 K^2 \langle P_{s,1}|Q_t|P_{s,2}] \langle P_{s,2}|Q_t|P_{s,1}]}{\langle P_{s,1}|Q_t|P_{s,2}]^2} \\ & + \{P_{s,1} \leftrightarrow P_{s,2}\} \Bigg\} \end{aligned} \tag{C.3}$$

For $n = 2$ the result is

$$C[Q_s, K]_{n=2} = \frac{(K^2)^3}{12\Delta_s^3} \left\{ \frac{\prod_{j=1}^{k+2} \langle P_{s,1}|R_j|P_{s,2}]}{\prod_{t=1,t\neq s}^{k} \langle P_{s,1}|Q_t|P_{s,2}]} (\mathcal{A}^3 + 3\mathcal{A}\mathcal{B} + \mathcal{C}) + \{P_{s,1} \leftrightarrow P_{s,2}\} \right\}, \tag{C.4}$$

where we have defined

$$\mathcal{A} = \sum_{j=1}^{k+2} \frac{(2Q_s \cdot K)(2R_j \cdot Q_s) - 2Q_s^2(2R_j \cdot K)}{\langle P_{s,1}|R_j|P_{s,2}]} - \sum_{t=1,t\neq s}^{k} \frac{(2Q_s \cdot K)(2Q_t \cdot Q_s) - 2Q_s^2(2Q_t \cdot K)}{\langle P_{s,1}|Q_t|P_{s,2}]}$$

$$\mathcal{B} = -\sum_{j=1}^{k+2} \frac{[(2Q_s \cdot K)(2R_j \cdot Q_s) - 2Q_s^2(2R_j \cdot K)]^2 + 2Q_s^2 K^2 \langle P_{s,1}|R_j|P_{s,2}] \langle P_{s,2}|R_j|P_{s,1}]}{\langle P_{s,1}|R_j|P_{s,2}]^2}$$
$$+ \sum_{t=1,t\neq s}^{k} \frac{[(2Q_s \cdot K)(2Q_t \cdot Q_s) - 2Q_s^2(2Q_t \cdot K)]^2 - 2Q_s^2 K^2 \langle P_{s,1}|Q_t|P_{s,2}] \langle P_{s,2}|Q_t|P_{s,1}]}{\langle P_{s,1}|Q_t|P_{s,2}]^2}$$

$$\mathcal{C} = \sum_{j=1}^{k+2} \frac{[(2Q_s \cdot K)(2R_j \cdot Q_s) - 2Q_s^2(2R_j \cdot K)]^3 - 3Q_s^2 K^2 \langle P_{s,1}|R_j|P_{s,2}] \langle P_{s,2}|R_j|P_{s,1}]}{\langle P_{s,1}|R_j|P_{s,2}]^2}$$
$$\frac{2[(2Q_s \cdot K)(2R_j \cdot Q_s) - 2Q_s^2(2R_j \cdot K)]}{\langle P_{s,1}|R_j|P_{s,2}]} - \sum_{t=1,t\neq s}^{k} \frac{2[(2Q_s \cdot K)(2Q_t \cdot Q_s) - 2Q_s^2(2Q_t \cdot K)]}{\langle P_{s,1}|Q_t|P_{s,2}]}$$
$$\frac{[(2Q_s \cdot K)(2Q_t \cdot Q_s) - 2Q_s^2(2Q_t \cdot K)]^2 - 3Q_s^2 K^2 \langle P_{s,1}|Q_t|P_{s,2}] \langle P_{s,2}|Q_t|P_{s,1}]}{\langle P_{s,1}|Q_t|P_{s,2}]^2}$$

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